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Low-temperature series expansions for the (2+1)-dimensional Ising model

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Abstract. Extended low-temperature series are calculated for the vacuum energy, magnetization, susceptibility and mass gap of the (2+1)-dimensional Ising model on the square lattice, triangular lattice and honeycomb lattice. A critical point analysis based upon them is presented.

Low-temperature series expansions for the (2+1)-dimensional Ising model, which is equivalent to the two-dimensional Ising model in a transverse field, have been calculated previously by Pfeuty and Elliott (1977), Yanase *et al* (1976) and Marland (1981). The model on a square lattice is also equivalent, via a duality transformation, to the Z_2 gauge model in 2+1 dimensions, for which series have been calculated by Irving and Hamer (1984) and Hamer and Irving (1985) Here we present some substantial extensions to the series on a square lattice and triangular lattice and a new series for the honeycomb lattice, and a critical point analysis based upon them.

To calculate the series, we used Nickel's (1980) cluster expansion methods, which have been discussed and extended by Marland (1981), Irving and Hamer (1984), and Hamer and Irving (1984). The techniques necessary were reviewed recently in He *et al* (1990), and will not be repeated here. The major difference is that a 'low-temperature' expansion is involved in the present case, requiring the calculation of 'strong' embedding constants for the clusters involved (Domb 1974).

The Hamiltonian of the model is (Fradkin and Susskind 1978, Suzuki 1976, Marland 1981)

$$H = \frac{1}{4}\sum_{\langle ij\rangle}(1-\sigma_i^z\sigma_j^z) + \lambda\sum_i \sigma_i^x + \frac{h}{2}\sum_i \sigma_i^z \qquad (1)$$

where the σ_i^α are Pauli spin operators acting on a 2-state spin variable at site i of the lattice, $\langle ij\rangle$ denotes nearest-neighbour pairs, λ corresponds to the temperature in the Euclidean formulation, and h is the magnetic field variable. The first term is taken as the unperturbed Hamiltonian, so the basis states are eigenstates of σ_i^z. The operator σ_i^x then acts as a perturbation, flipping the spin on site i.

Series have been calculated for the ground-state energy per site E_0/N, the magnetization $M = (1/N)\,\partial E_0/\partial h\,|_{h=0}$, the susceptibility $\chi = -(1/N)\,\partial^2 E_0/\partial h^2\,|_{h=0}$, and

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Table 1. Low-temperature series in $x = \lambda^2$ for the vacuum energy per site E_0/N, the magnetization M, the susceptibility χ and the mass gap m. Coefficients of x^n are listed.

n	E_0/N	M	χ	m
Square lattice				
0	0	1/2	0	2
1	−1/2	−1/4	1/4	−3
2	−0.416666666667E−01	−0.173611111111E+00	0.606481481481E+00	0.358333333333E+01
3	−0.104166666667E−01	−0.146701388889E+00	0.108463541667E+01	−0.231400462963E+02
4	−0.130931712963E−01	−0.202285879630E+00	0.222461287133E+01	0.133219982960E+03
5	−0.557857912166E−02	−0.225083295356E+00	0.387519301630E+01	−0.849049962874E+03
6	−0.107881014460E−01	−0.348168014525E+00	0.746815535149E+01	0.573800145770E+04
7	−0.447580084560E−02	−0.416850655204E+00	0.126701858645E+02	−0.405732511684E+05
8	−0.149426018529E−01	−0.719440808188E+00	0.242430294830E+02	0.296147498339E+06
9	−0.537214550564E−02	−0.895896725296E+00	0.409644988269E+02	−0.221568466229E+07
10	−0.205225529462E−01	−0.153803037929E+01	0.757231701094E+02	0.169055794691E+08
11	−0.137135122388E−01	−0.215114454517E+01	0.131409431703E+03	
12	−0.285910551200E−01	−0.346738008835E+01	0.234957352434E+03	
13	−0.279709766479E−01	−0.517153576700E+01	0.410148720685E+03	
Triangular lattice				
0	0	1/2	0	3
1	−1/3	−1/9	2/27	−4/3
2	−0.740740740741E−02	−0.251851851852E−01	0.596543209877E−01	−0.296296296296E−00
3	−0.159905937684E−02	−0.107437081829E−01	0.487608425616E−01	0.564750146972E−01
4	−0.403399716409E−03	−0.507865259250E−02	0.373911240328E−01	−0.461890116036E−01
5	−0.137731238619E−03	−0.263892944508E−02	0.280504999324E−01	−0.190642432806E−01
6	−0.535575784907E−04	−0.144521904116E−02	0.207221811427E−01	−0.195428063630E−01
7	−0.248710211255E−04	−0.838930519777E−03	0.153204359589E−01	0.386611895940E−01
8	−0.107356601094E−04	−0.487856529964E−03	0.111475531172E−01	−0.250972109091E−01
9	−0.541667159130E−05	−0.293767274739E−03	0.809911081490E−02	−0.273224962444E−01
10	−0.288207996041E−05	−0.181247457250E−03	0.588750688243E−02	
11	−0.139397178749E−05	−0.110875060413E−03	0.423467828220E−02	
Honeycomb lattice				
0	0	1/2	0	3/2
1	−2/3	−4/9	16/27	−20/3
2	−0.148148148148E+00	−0.740740740741E+00	0.345679012346E+01	0.411851851852E+02
3	−0.105349794239E+00	−0.150474622771E+01	0.147082359396E+02	−0.669181893004E+03
4	−0.110909922268E+00	−0.335010394757E+01	0.553997838541E+02	0.136839672977E+05
5	−0.139683546111E+00	−0.786783907538E+01	0.196032331298E+03	−0.319813937454E+06
6	−0.597139616123E+00	−0.256464256931E+02	0.779926148575E+03	0.799694187477E+07
7	−0.111213720360E+01	−0.747015811873E+02	0.296482744422E+04	−0.209446650354E+09
8	−0.163003207836E+01	−0.201987675737E+03	0.105783734100E+05	0.567514343659E+10
9	−0.244796899957E+01	−0.544219703853E+03	0.364689040955E+05	−0.157786392106E+12
10	−0.225570897762E+02	−0.205593247077E+04	0.141568686503E+06	0.447623010440E+13
11	−0.384355319659E+02	−0.617694765252E+04	0.517165348132E+06	
12	−0.408320974454E+02	−0.169554762563E+05	0.178505968289E+07	
13	−0.236973596439E+03	−0.554334081611E+05	0.639727954765E+07	
14	−0.127601736068E+04	−0.204618496991E+06	0.241744275384E+08	

the mass gap m, as listed in table 1. From the ground-state energy, one can derive the 'specific heat' $C = -(1/N)\partial^2 E_0/\partial y^2$, where $y = \lambda^{-1}$ (Marland 1981). For the ground-state energy and its derivatives, a list of 8740 linked clusters (up to 13 sites) for the square lattice or 6634 linked clusters (up to 11 sites) for the triangular lattice or 2265 linked clusters (up to 14 sites) for the honeycomb lattice was required. The results agree with the earlier calculation of Marland (1981), and Irving and Hamer (1984), and add several terms to each series. The mass gap involved 3054 clusters (up to 11 sites) for the square lattice or 4302 clusters (up to 10 sites) for the triangular lattice or 874 clusters (up to 11 sites) for the honeycomb lattice, both linked and unlinked. The results for the square lattice agree with those of Hamer and Irving (1985), and add three more terms. Generating the cluster data occupied some 80 hours of CPU time on an IBM3090, while calculating the contribution of each cluster to the series took up 90 hours of CPU time for the ground-state energy and its derivatives, and 83 hours of CPU time for the mass gap

Table 2. Dlog Padé approximants to the magnetization series of the square lattice. An asterisk denotes a defective approximant.

	$[N/(N-1)]$		$[N/N]$		$[N/(N+1)]$	
N	Pole	Residue	Pole	Residue	Pole	Residue
2	0.5343	(0.233)	0.5727	(0 298)	0.5734	(0.299)
3	0.5744	(0.301)	0.5718	(0.296)*	0.5743	(0.300)*
4	0.5779	(0.310)	0 5768	(0.307)*	0 5810	(0.331)*
5	0.5775	(0.309)	0.5780	(0.311)	0.5783	(0.313)
6	0.5796	(0 324)*	0.5803	(0.333)*		

To determine the critical point parameters, the series have been analysed using standard Dlog Padé approximant methods (Guttmann 1989). The best behaved series for the square lattice is that for the magnetization: results from the $[N/M]$ Dlog Padé approximants to this series are given in table 2. From this table, we estimate that the critical point lies at $x_c = 0.5780(10)$, where $x = \lambda^2$, with index $\beta = 0\ 311(4)$. Trends in the table indicate that these results may slightly underestimate the true values. Now the most accurate estimate of the critical point for this model is $x_c = 0.57917(14)$ for the square lattice and $x_c = 1.4210(3)$ for the triangular lattice, derived from the high-temperature series of He *et al* (1990) Taking this as the correct value (for the honeycomb lattice, we choose $x_c = 0.2830(7)$ as the correct value), biased Dlog Padé approximants give the magnetization exponent for the square lattice as 0.318(4). We have also tried analysing the series using differential approximants (Guttmann 1989), but no significant improvement in the results was obtained. Using biased confluent differential approximants, a slight improvement can be achieved in some cases.

The other series were analysed in a similar way, and the resulting estimates of the critical parameters are listed in table 3, along with some selected results from other sources for comparison. The coefficients in the mass gap series of the square lattice alternate in sign, and the series is dominated by a non-physical singularity at $x \approx -1/7$. We tried to remove the effect of this by an Euler transformation, but again no great improvement was observed.

As can be seen from table 3, the estimates of γ and ν (or more strictly, γ' and ν') from the low-temperature series are consistent with those from high-temperature

Table 3. A comparison of results obtained in the present work with some previous calculations. Key: SQ, square lattice; TR, triangular lattice; HC, honeycomb lattice, HT, high-temperature series; LT, low-temperature series; FS, finite-size scaling; MC, Monte Carlo; (ub), unbiased; (b), biased.

		γ	β	ν	α	x_c
(2+1)-dimensional Ising model						
FS SQ	Henkel (1990)		0.324(9)	0.629(2)	0.11(1)	0.581(2)
HT SQ	He *et al* (1990)	1.244(3)		0.638(2)	0.11(2)	0.57917(14)
LT SQ	Marland (1981)	1.25(2)	0.315(5)		0.097(1)	0.579(1)
LT SQ	This work (ub)	1.28(3)	0.311(4)	0.6(2)	0.11(4)	0.578(1)
LT SQ	This work (b)	1 25(2)	0.318(4)	0.64(3)	0.096(6)	
HT TR	He *et al* (1990)	1.241(3)		0.636(4)	0.10(2)	1.4210(3)
LT TR	Marland (1981)	1.250(12)	0.315(2)		0.098(3)	1.420(1)
LT TR	This work (ub)	1.28(2)	0 315(8)	0 62(10)	0.108(14)	1.423(2)
LT TR	This work (b)	1.255(10)	0.320(3)	0 64(1)	0.096(8)	
LT HC	This work (ub)	1.3(2)	0.307(10)	0.6(4)	0.13(4)	0.2830(7)
LT HC	This work (b)	1.25(2)	0.310(7)	0.6(2)	0.106(6)	
Three-dimensional Ising model						
HT	Guttmann (1987)	1.239(3)		0.632(3)		
HT	Camp *et al* (1976)				0.125(20)	
LT	Gaunt *et al* (1973)		0 312(5)			
MC	Bhanot *et al* (1987)			0.6295(10)		
Field-theory estimate						
	Le Guillou *et al* (1980)	1.241(2)	0.325(2)	0.630(2)		

series, but are less accurate by almost an order of magnitude. The estimates for β and α (or α'), however, are competitive with those from other sources. Our accuracy is no better than that claimed by Marland (1981), even though he worked with shorter series: this may be because he used the Rushbrooke scaling relation

$$\alpha' + 2\beta + \gamma' = 2 \tag{2}$$

as a constraint. We note in passing, however, that our results in table 3 satisfy the scaling relation better than the final estimates of Marland.

In any case, our results in table 3 are consistent, within errors, with the accepted ideas of universality between the Euclidean and Hamiltonian forms of the Ising model, and equivalence between low-temperature and high-temperature critical exponents.

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